Coordination Problem

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1 Preliminaries

If you decide to take a course in First-Order Logic, you might see some of these terms and symbols. If you haven't, let me illuminate you for the purposes of this paper. But be warned, this requires active reading. I recommend a paper and pencil. We say that x is a "variable". We may replace x with any object. Rx is a statement R about x. For example, x is red or x is prime. Rxy is a statement about both x and y. For example, x loves y. An interpretation I is the universe dictating the rules of how statements and variables interact. For example, a healthy person has highlighter yellow skin might be true in the Simpsons universe but this is clearly not true in ours. The symbol "$\forall x$" is read as "for all x". For example, you can read $\forall x Rxx$ can be read as "For all x, x loves x." The symbol "$\exists x$ is read as "there exists x". For example, you can read $\exists x Rxx$ can be read as "There exists x such that x loves x." We say that o_n,where n is a positive whole number is specific value of a variable. So when you read val$a = \{o_1, o_2, \ldots o_n\}$, this means that within our universe I a can only take in the values of $o_1, o_2, \ldots o_n$. For example, let Ra be the statement "a is prime." Then a can only take on certain values. We extend this idea to something called the universal object, denoted as $\underline{o}$, which may take on any value in the universe I. We say $x \in R$ if Rx is a true statement in I. Similarly, we say $x \notin R$ if Rx is a false statement in I. When we write $Aw, Bx, Cy \models Dz$,we say "The statement Dz is a logical consequence of the statements Aw, Bx, Cy." Similarly, $Aw, Bx, Cy \not\models Dz$ is read as, "The statement Dz is not a logical consequence of the statements Aw, Bx, Cy." $\forall_x$ is the truth law governing how $\forall x$ and Rx interact with each other. If Rx is true in I, then we may replace x with the universal object. If it is false, then we may replace the statement $\forall x, Rx$ with $\exists x$, not Rx. The idea behind truth law is that if $\forall x, Rx$ is true, then it does not matter what values x takes on. If it is false, then we need to find just one counterexample.

That was a mouthful. But first and second order logic serve as the logical basis for modern math. This idea is important for making math as sound as it is. I recommend playing with these ideas to really understand it. The following problem was proposed in my own logic course and it took me a while to figure out the solution. This is a good example in demonstrating the subtitles of logic. The question arises from showing $\forall x, Rxx \not\models Rab$, which is a true statement. (It is an exercise for the reader to understand why.) However, when constructing a proof by construction by assuming $\forall x, Rxx \not\models Rab$ by assuming Rxx is true and Rab is false (which is what it means for something to not be a logical consequence), we observe that those statements lead to contradictions and therefore, $\forall x, Rxx \models Rab$.

2 Statement of the problem

Suppose we are asked to show $\forall x, Rxx \not\models Rab$ by means of a semantic proof. The false proof is given as follows:

Proof. Suppose there is an interpretation I for $\forall x, Rxx \not\models Rab$ such that :

(1) $\forall x, Rxx$ is true in I

(2) Rab is false in I

(1.1) Rxx is true in I for any variable assigned to x by (1) and $\forall_T$.

(1.2) $(\underline{o}, \underline{o}) \in R$ by (1.1).

(2.1) If val$a = \{o_1\}$ and val$b = \{o_2\}$ in I, then $(o_1, o_2) \notin R$, by (2).

(1.3) Since $\underline{o}$ is a universal object, we may replace $(\underline{o}, \underline{o}) \in R$ with $(o_1, \underline{o}) \in R$ by (1.2)

(1.4) We may apply this rule again in (1.3) to a achieve the result $(o_1, o_2) \in R$

We have a contradiction with (1.4) and (2.1) so then $\forall x, Rxx \models Rab$

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3 Where the problem is + my solution

Before we tackle $\forall x, Rxx \models Rab$, let us look at an easier version of this problem. Namely, $\forall x, \forall y Rxy \models Rab$. It is easy to see that the above proof will hold. Why?

Suppose we let $D = \{o_1, o_2, o_3, \ldots, o_n\}$. As soon as we fix x to be some variable, say o_i, we are allowed to pick any of the remaining variables within our domain for y. In other words, as soon as we pick (o_i, y) for our x value, we may pick any value for y that does not depend on our choice of x. This differs from $\forall x, Rxx \models Rab$ because as soon as we fix the first coordinate for some $(0_i, x) \in R$. We can pick one and only one element from our domain for the 2nd coordinate, which in this case happens to be what ever we picked for the first one.

The idea of choosing the 2nd coordinate based on the choice of the first is not a new idea. This is just a function, which has the following definition in QL.

$$\forall x \in X, \exists y \in Y \text{ such that } (x, y) \in R \text{ where}$$

How do we apply this to our problem? We can think as the 2nd x in $(x, x) \in R$ as some function $y(x)$ which happens to map a value to itself. More precisely, $y(x) : o_j \mapsto o_j$, for some $o_j \in D$ So we should rewrite our problem as $\forall x, Rx, y(x) \not\models Rab$, where $y(x)$ is defined as before.

Even though it isn't *exactly* the same problem, it captures the same idea without the contradiction we get earlier. Anything true for Rxx will be true for $Rx, y(x)$. Putting everything together we have the final statement. $\forall x \exists y(x)$